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PERIODIC MOTIONS OF A RIGID BODY WITH A FIXED POINT IN A NEWTONIAN FIELD

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Poincaré's method of small parameter was used in [1] to establish the existence of periodic motions of a rigid body with a fixed point in a Newtonian force field. It was assumed that the body was nearly dynamically symmetric and its center of gravity close to the fixed point. The canonical Andoyer elements were used as the independent variables. The free rotation of the axisymmetric rigid body was taken as the unperturbed motion [2].

In the present paper the existence of periodic motions of a rigid body, fixed at the center of mass, in a central Newtonian force field, is investigated. The generating solution corresponds to a free Euler-Poinsot rotational motion, and the canonical action-angle variables are taken as the independent variables. It is assumed that the inertia ellipsoid of the body is nearly spherical.

1. We use the canonical action-angle: L, G, H, l, g, h [3] to describe the motion of a rigid body about a fixed center of mass in the Newtonian gravity field. The Hamiltonian $F = T - U$ of the problem is given in terms of these variables by the formulas

$$T = \frac{1}{2D}\bar{L}^2 + \frac{1}{4}\left(\frac{1}{A} + \frac{1}{B}\right)\bar{G}^2, \quad \bar{L} = L\left[1 + \frac{1}{16}(b-1)(b+3)e^2 + \ldots\right] \tag{1.1}$$

$$\bar{G} = G, \quad b = \frac{G^2}{L^2}, \quad \frac{1}{D} = \frac{1}{C} - \frac{1}{2}\left(\frac{1}{A} + \frac{1}{B}\right), \quad e = \frac{1}{2}\left(\frac{1}{B} - \frac{1}{A}\right)D$$

$$U = 3P/2g_0R\,(A\gamma^2 + B\gamma'^2 + C\gamma''^2)$$

Here A, B and C are the principal central moments of inertia of the body, P denotes the weight of the body, g_0 the acceleration due to gravity and γ, γ', γ'' are the direction cosines of the radius-vector R of the center of mass originating at the center of attraction in the coordinate system rigidly bound with the body.

In what follows, we must obtain an expression for U in terms of the action-angle variables. To this end we employ the formulas describing the unperturbed Eulerian motion [3]. After some transformations, we obtain the following expression for U in the form of a series:

$$U = (B - A)\frac{3P}{2g_0R}\sum_{k_1,k_2} U_{k_1k_2}\left(\frac{L}{G}, \frac{H}{G}\right)\cos(k_1 l + k_2 g) \tag{1.2}$$

$$(k_1 = 0, 2, 4, \ldots; \quad k_2 = -2, -1, 0, 1, 2)$$

$$U_{0,0} = {}^1/_4 \sin^2\rho\,(2\delta - 1) + {}^1/_2\,(1 - {}^3/_2 \sin^2\rho)\,[(2\delta - 1)\cos^2\theta\, d_{00}^\circ + \sin^2\theta\, d_{00}^2]$$

$$U_{2m,0} = {}^1/_2\,(1 - {}^3/_2 \sin^2\rho)\,[(2\delta - 1)\cos^2\theta\, d_{0,m}^\circ + \sin^2\theta\, d_{0,m}^2]$$

$$U_{2m,\pm1} = {}^1/_4 \sin 2\rho\,(1 - 2\delta)\sin 2\theta\, d_{\pm1,m}^\circ \pm {}^1/_4 \sin^2\rho \sin\theta\, d_{\pm1,m}^2$$

$$U_{2m,\pm2} = -{}^1/_4 \sin^2\rho\,(1 - 2\delta)\sin^2\theta\, d_{\pm2,m}^\circ + {}^1/_8 \sin^2\rho\,(1 \pm \cos\theta)^2 d_{\pm2,m}^2$$

$$\cos\rho = H / G, \quad \cos\theta = L / G, \quad \delta = (C - A)/(B - A)$$

The quantities $d^k_{n,m}$ are given by the well known series in increasing powers of e, the coefficients of which are functions of θ. Thus the problem can be described by the following canonical expressions:

$$dL / dt = -\partial F / \partial l, \quad dl / dt = \partial F / \partial L \qquad (1.3)$$
$$dG / dt = -\partial F / \partial g, \quad dg / dt = \partial F / \partial G$$
$$dH / dt = -\partial F / \partial h, \quad dh / dt = \partial F / \partial H$$

The above system admits two first integrals

$$F = c_1, \quad H = c_2 \qquad (1.4)$$

We now assume that the inertia ellipsoid of the body is nearly spherical. Then choosing the quantity $\nu = 3P\,(B - A)\,/\,(2g_0R)$ as the small parameter, we transform the Hamiltonian F to the form necessary for the application of the Poincaré method of small papameter

$$F = F_0(L, G) + \nu F_1\left(\frac{L}{G}, \frac{H}{G}, l, g\right), \quad F_0 = T, \quad \nu F_1 = -U \qquad (1.5)$$

2. Let us introduce new notation for the action-angle variables

$$x_1 = L, \quad x_2 = G, \quad x_3 = H, \quad y_1 = l, \quad y_2 = g, \; y_3 = h$$

Then we can write the equations of motion in the form

$$dx_i / dt = -\partial F / \partial y_i, \quad dy_i / dt = \partial F / \partial x_i \quad (i = 1, 2, 3) \qquad (2.1)$$

where (1.5) now becomes

$$F = F_0(x_1, x_2) + \nu \sum_{k_1, k_2} U_{k_1k_2}(x_1, x_2, x_3) \cos(k_1y_1 + k_2y_2) \qquad (2.2)$$

When $\nu = 0$, (2.1) yields the following generating system of equations:

$$dx_i/dt = 0, \quad dy_i/dt = \partial F_0/dx_i = n_i^{(0)}$$

the general solution of which has the form

$$x_i^{(0)} = a_i, \quad y_i^{(0)} = n_i^{(0)}\, t + \omega_i \quad (i = 1,2,3) \qquad (2.3)$$

where a_i, ω_i are arbitrary constants of integration and (with (1.2) taken into account)

$$n_1^{(0)} = \left.\frac{\partial F_0}{\partial x_1}\right|_{x_i=a_i} = \frac{a_1}{D}\left[1 - \frac{e^2}{8}(b_0^2 - 3) + \ldots\right] \qquad (2.4)$$
$$n_2^{(0)} = \left.\frac{\partial F_0}{\partial x_2}\right|_{x_i=a_i} = \frac{1}{2}\left(\frac{1}{A} + \frac{1}{B}\right)a_2 + \frac{a_1}{4D}\,[b_0^{1/2}\,(b_0 + 1)\,e^2 + \ldots]$$
$$n_3^{(0)} = \left.\frac{\partial F_0}{\partial x_3}\right|_{x_i=a_i} = 0, \quad b_0 = \frac{a_1^2}{a_2^2}$$

The above solution will be T-periodic provided that the following conditions hold for the integers $\bar{k}_i$, $(i = 1, 2, 3)$:

$$x_i\,(T) - x_i\,(0) = 0, \quad y_i\,(T) - y_i\,(0) = n_i^{(0)}\,T = 2\bar{k}_i\pi$$

We consider a solution of (2.1) with initial conditions $x_i = a_i + \beta_i$, $y_i = \omega_i + \gamma_i$, which we shall write in the form

$$x_i = a_i + \beta_i + \xi_i(t), \quad y_i = n_i^{(0)} t + \omega_i + \gamma_i + \eta_i(t) \quad (i = 1, 2, 3) \tag{2.5}$$

Using the formulas (2.5) we pass to the new variables ξ_i, η_i. The equations of motion now become

$$d\xi_i / dt = -\partial K / \partial\eta_i, \quad d\eta_i / dt = \partial K / \partial\xi_i \quad (i = 1, 2, 3) \tag{2.6}$$

$$K = \xi_1 n_1^{(0)} + \xi_2 n_2^{(0)} + \sum_{i=1}^{3}\left[\frac{\partial F}{\partial a_i}\xi_i + \frac{\partial F}{\partial \omega_i}\eta_i\right] +$$

$$\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left[\xi_i\xi_j\frac{\partial^2 F}{\partial a_i \partial a_j} + \xi_i\eta_j\frac{\partial^2 F}{\partial a_i \partial \omega_j} + \eta_i\eta_j\frac{\partial^2 F}{\partial \omega_i \partial \omega_j}\right] + O(\xi^3)$$

The necessary and sufficient conditions for the periodic solutions of the system (2.6) to exist, are

$$\xi_i(T) = 0, \quad \eta_i(T) = 0 \quad (i = 1, 2, 3) \tag{2.7}$$

Expanding $F(a_i + \beta_i \mid n_i^{(0)} t + \omega_i + \gamma_i)$ into a power series in β, γ, ν, we obtain from (2.6) the following explicit expressions for the conditions (2.7):

$$-\frac{\xi_k(T, \beta, \gamma, \nu)}{\nu T} = \frac{\partial [F_1]}{\partial \omega_k} + \sum_{j=1}^{2}\left\{\frac{\partial^2 [F_1]}{\partial \omega_k \partial a_j}\beta_j + \right. \tag{2.8}$$

$$\left.\frac{\partial^2 [F_1]}{\partial \omega_k \partial \omega_j}\gamma_j\right\} + \ldots = 0 \quad (k = 1, 2)$$

$$\xi_3(T, \beta, \gamma, \nu) = 0$$

$$\frac{\eta_1(T, \beta, \gamma, \nu)}{T} = \beta_1\frac{\partial^2 F_0}{\partial a_1^2} + \beta_2\frac{\partial^2 F_0}{\partial a_1 \partial a_2} + \nu\left\{\frac{\partial [F_1]}{\partial a_1} - \right.$$

$$\frac{1}{T}\int_0^T \frac{\partial^2 F_0}{\partial a_1^2}\int_0^t \frac{\partial F_1}{\partial \omega_1}\xi d\xi dt + \beta_1\left(\frac{\partial^2 [F_1]}{\partial a_1^2} - \frac{1}{T}\frac{\partial^2 F_0}{\partial a_1^2}\int_0^T \frac{\partial^2 F_1}{\partial a_1 \partial \omega_1} t dt\right) +$$

$$\left.\beta_2\frac{\partial^2 [F_1]}{\partial a_1 \partial a_2} + \beta_3\frac{\partial^2 [F_1]}{\partial a_1 \partial a_3} + \gamma_1\frac{\partial^2 [F_1]}{\partial a_1 \partial \omega_1} + \gamma_2\frac{\partial^2 [F_1]}{\partial a_1 \partial \omega_2}\right\} + \ldots = 0$$

$$\frac{\eta_2(T, \beta, \gamma, \nu)}{T} = \beta_1\frac{\partial^2 F_0}{\partial a_2 \partial a_1} + \beta_2\frac{\partial^2 F_0}{\partial a_2^2} + \nu\left\{\frac{\partial [F_1]}{\partial a_2} - \frac{1}{T}\int_0^T \frac{\partial^2 F_0}{\partial a_2^2}\int_0^t \frac{\partial F_1}{\partial \omega_1} \times \right.$$

$$\xi d\xi dt + \beta_1\frac{\partial^2 [F_1]}{\partial a_2 \partial a_1} + \beta_2\left(\frac{\partial^2 [F_1]}{\partial a_2^2} - \frac{1}{T}\frac{\partial^2 F_0}{\partial a_2^2}\int_0^T \frac{\partial^2 F_1}{\partial a_2 \partial \omega_1} t dt\right) +$$

$$\left.\beta_3\frac{\partial^2 [F_1]}{\partial a_2 \partial a_3} + \gamma_1\frac{\partial^2 [F_1]}{\partial a_2 \partial \omega_1} + \gamma_2\frac{\partial^2 [F_1]}{\partial a_2 \partial \omega_2}\right\} + \ldots = 0$$

$$\frac{\eta_3(T, \beta, \gamma, \nu)}{\nu T} = \frac{\partial [F_1]}{\partial a_3} + \beta_1\left(\frac{\partial^2 [F_1]}{\partial a_3 \partial a_1} - \frac{1}{T}\frac{\partial^2 F_0}{\partial a_1^2}\int_0^T \frac{\partial^2 F_1}{\partial a_3 \partial \omega_1} t dt\right) +$$

$$\beta_2\frac{\partial^2 [F_1]}{\partial a_3 \partial a_2} + \beta_3\frac{\partial^2 [F_1]}{\partial a_3^2} + \gamma_1\frac{\partial^2 [F_1]}{\partial a_3 \partial \omega_1} + \gamma_2\frac{\partial^2 [F_1]}{\partial a_3 \partial \omega_2} + \ldots = 0$$

$$[F_1] = \frac{1}{T}\int_0^T F_1(a_i \mid n_i^{(0)} t + \omega_i)\, dt$$

The equations of motion (2.1) admit two first integrals (1.4), the Jacobian of which, written in the generating solution in terms of the variables x_1, x_3 is different from zero

$$\left.\frac{\partial (c_1, c_2)}{\partial (x_1, x_3)}\right|_{x_i=a_i} = n_1^{(0)} \neq 0 \tag{2.9}$$

Consequently the first and third condition of (2.8) must be regarded as being derivable from the remaining four conditions. We therefore assume that $\xi_1 = \xi_3 = 0$, considering the following four equations only:

$$\xi_2 = \eta_1 = \eta_2 = \eta_3 = 0 \tag{2.10}$$

In this manner we obtain four equations for six unknowns β, γ. It follows that two unknowns (say γ_1 and γ_3) can be assumed arbitrary. Let $\gamma_1 = \gamma_3 = 0$ and let us choose the reference point at the initial instant of time so that $\omega_1 = \omega_3 = 0$. Then Eqs. (2.10) will yield β_1, β_2, β_3, γ_2 as holomorphic functions of ν, vanishing together with ν if the conditions [4]

$$\frac{\partial [F_1]}{\partial \omega_2} = 0, \quad \frac{\partial [F_1]}{\partial a_3} = 0, \quad \frac{\partial (\xi_2, \eta_1, \eta_2, \eta_3)}{\partial (\beta_1, \beta_2, \beta_3, \gamma_2)} \neq 0 \tag{2.11}$$

are satisfied.

The equations $\eta_1 = 0$ and $\eta_2 = 0$ depend only on β_1 and β_2 when $\nu = 0$. Consequently the Jacobian of (2.11) is a product of the Jacobian of η_1 and η_2 in β_1 and β_2 which represents a Hessian F_0 in x_1 and x_2 in the generating solution, and the Jacobian of ξ_2 and η_3 in γ_2 and β_3 which is equal to the Hessian $[F_1]$ in a_3 and ω_2.

Thus the third condition of (2.11) is equivalent to the other two conditions

$$H(F_0)\Big|_{\substack{x_1=a_1\\x_2=a_2}} \neq 0, \quad H([F_1])\Big|_{\substack{a_3\\\omega_2}} \neq 0 \tag{2.12}$$

3. Let us now return to the previous action-angle variables and consider the conditions of existence of periodic solutions. The first condition of (2.12) is satisfied at all times except the case when $1/D = 0$, since we have, with the accuracy to within e^2,

$$H(F_0) = \frac{1}{2D}\left(\frac{1}{A} + \frac{1}{B}\right) + \frac{e^2}{8D^2} L_0 b_0^{-1/2} (3b_0 + 1) + \ldots \neq 0$$

Before considering the remaining conditions, let us obtain the explicit expression for $[F_1]$:

a) in the case of commensurability $Nn_1^{(0)} = n_2^{(0)}$ we have

$$[F_1] = U_{0,0}\left(\frac{L_0}{G_0}, \frac{H_0}{G_0}\right) + U_{2N,-2}\left(\frac{L_0}{G_0}, \frac{H_0}{G_0}\right) \cos 2(Nl_0 - g_0)$$

b) in the case of commensurability $2Nh_1^{(0)} = n_2^{(0)}$ we have

$$[F_1] = U_{0,0}\left(\frac{L_0}{G_0}, \frac{H_0}{G_0}\right) + U_{2N,-1} \cos (2Nl_0 - g_0) + U_{4N,-2} \cos 2(2Nl_0 - g_0)$$

Here N is a positive integer and the coefficients $U_{0,0}$, $U_{2N,-2}$, $U_{4N,-2}$, $U_{2N,-1}$ are computed from (1.2) using the generating values of the variables L_0, G_0 and H_0. Substituting the expression for $[F_1]$ into the first condition of (2.11) we obtain, in both cases, for the angular variables l, g, h the following generating values $l_0 = 0$; $g_0 = 0$, $\pi/2$, π, $^3/_2\pi$; $h_0 = 0$.

The second condition of periodicity in (2.11) can be conveniently written in terms of a new quantity ρ. Equation (2.11) then becomes

$$\frac{1}{\sin\rho}\frac{\partial[F_1]}{\partial\rho}=0$$

In the case (a) of commensurability this condition becomes

$$\begin{aligned}&\cos\rho\,\{(2\delta-1)-3\,[(2\delta-1)\cos^2\theta d_{0,0}^{\circ}+\sin^2\theta d_{00}{}^2]+\\&\quad p\,[(2\delta-1)\sin^2\theta d_{-2,2N}^{\circ}+{}^1/_2\,(1-\cos\theta)^2 d_{2,2N}^2]\}=0\\&(p=\cos 2g_0=\pm 1)\end{aligned}\tag{3.1}$$

The condition (3.1) folds if $\rho=\pi/2$ and the quantities δ, e and θ can assume arbitrary values. If the arbitrary value falls within the interval $(0, \pi/2)$, then the quantities δ, e, θ are connected by some relation.

Introducing new parameters ε and μ and characterizing the deviation of the inertia ellipsoid of the body from the spherical shape by the formulas $B = A\,(1+\varepsilon)$, $C = A\,(1+\mu)$, we arrive at an equation of the type $f(\varepsilon, \mu, \theta) = 0$, which can be analyzed by numerical methods. Finally, the last condition of existence of periodic solutions almost equal to the generating solutions obtained always holds. It can be written in an explicit form and verified directly.

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